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Some New Recursion Relations for Spline Functions with Application to Spline Collocation*

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In this paper we treat two special Hermite–Birkhoff interpolation problems in the space of spline functions and develop some recursion relations for the calculation of the solutions of these interpolation problems. Furthermore we show that these formulas can be used for the continuous approximation of the solution of a nonlinear two-point boundary value problem.

1. INTRODUCTION

Let $[a, b]$ be a finite real interval, N, m positive integers, $m \geqslant 3$, $h = (b-a)/(N+1)$, and $x_i := a + ih$ $(i = 0, ..., N+1)$. Denote by $S_m(\Delta)$ the space of polynomial spline functions of degree m with simple knots x_i $(i = 1, ..., N)$. The B-splines associated with the given knot partition Δ are defined by

$$B_{m,i}(x) := \frac{1}{h^m \cdot m!} \sum_{\nu=0}^{m+1} (-1)^\nu \binom{m+1}{\nu} (x - x_{i+\nu})_+^m.$$

In various topics of numerical analysis the following Hermite–Birkhoff interpolation problem arises. Given real data y_i, M_i $(i = 0, ..., N+1)$, a spline function s is looked for such that

$$\begin{aligned} & s \in S_m(\Delta) \\ & s(x_i) = y_i \qquad (i = 0, ..., N+1). \\ & s''(x_i) = M_i. \end{aligned} \tag{1.1}$$

* This paper is part of my doctoral thesis [4].

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The interpolation problem (1.1) is not uniquely solvable for all $m \in \mathbb{N}$. For example, if m is even then

$$s_0(x) = \sum_{i=-m}^{N} (-1)^i B_{m,i}(x) \tag{1.2}$$

is a nontrivial nullspline, i.e., $s_0(x_i) = s_0''(x_i) = 0$ $(i = 0, ..., N+1)$.

Therefore we also consider the problem

$$\begin{aligned} & s \in S_m(\Delta) \\ & s(x_i) = y_i, \qquad s''(x_i) = M_i \qquad (i = 0, ..., N+1) \\ & s^{(m-1)}(x_0) = y_0^{(m-1)}, \end{aligned} \tag{1.3}$$

where y_i, M_i, and $y_0^{(m-1)}$ are given real data.

In Section 2 we give necessary and sufficient conditions such that problem (1.1) ((1.3) respectively) is solvable. The main purpose of this paper is to develop some recursion relations which are useful in order to construct the solutions of (1.1) and (1.3), respectively. These recursion relations generalize the corresponding formulas of Usmani [7] and Usmani and Warsi [8], who have developed it for the case $m = 5$.

2. On the Solvability of a Special Hermite–Birkhoff Problem

We introduce the following notations:

$$b_{m,l}^{(\nu)} := B_{m,i}^{(\nu)}(x_{i+1}), \qquad b_{m,l} := b_{m,l}^{(0)},$$

$$(\nu = 0, ..., m-1;\ i = -m, ..., N;\ l = 0, ..., m+1)$$

$$a_l^{(m)} := (m-2)!\, h^2 b''_{m,l+1} \tag{2.1}$$

$$c_l^{(m)} := m!\, b_{m,l+1} \tag{2.2}$$

$$s := \begin{cases} (m-1)/2 & \text{if } m \text{ is odd} \\ (m-2)/2 & \text{if } m \text{ is even.} \end{cases} \tag{2.3}$$

The polynomials p_m, $q_m \in \pi_{2s}$ are defined by

$$p_m(x) := \sum_{l=0}^{2s} c_l^{(m)} x^l \tag{2.4}$$

$$q_m(x) := \sum_{l=0}^{2s} a_l^{(m)} x^l. \tag{2.5}$$

Note that $a_l^{(m)} = c_l^{(m-2)} - 2c_{l-1}^{(m-1)} + c_{l-2}^{(m-2)}$, hence $q_m(x) = (x-1)^2 P_{m-2}(x)$. In [4] we were able to show that all zeros of q_m (and thus all zeros of p_m)

are real. Denote by $P_m=\{\mu_0(m), ..., \mu_{2s-1}(m)\}$ and $Q_m=\{1, \lambda_2(m), ..., \lambda_{2s-1}(m)\}$ the set of zeros of p_m and q_m, respectively. We remark that -1 is a common zero of p_m and q_m if m is even.

The next theorem gives criterions for the solvability of the Birkhoff problems (1.1) and (1.3):

THEOREM 2.1. (1) *Let $m\geqslant 3$ be odd and so chosen that $P_m\cap Q_m=\varnothing$ and $N\geqslant m-3$. Then the Hermite–Birkhoff problem* (1.1) *is uniquely solvable, if and only if*

$$\sum_{l=1}^{m} b_{m,l}M_{i+l}=\sum_{l=1}^{m} b''_{m,l}y_{i+l} \qquad (i=-1, ..., N-m+1).$$

(2) *Let $m\geqslant 4$ be even, $P_m\cap Q_m=\{-1\}$, and $N\geqslant m-4$. Then problem* (1.3) *is uniquely solvable if and only if*

$$\sum_{\mu=1}^{m-1}\sum_{\sigma=1}^{\mu}(-1)^{\mu-\sigma}b''_{m,\sigma}y_{i+\mu}=\sum_{\mu=1}^{m-1}\sum_{\sigma=1}^{\mu}(-1)^{\mu-\sigma}b_{m,\sigma}M_{i+\mu}$$
$$(i=-1, ..., N-m+2).$$

A proof of this theorem can be found in [4, pp. 54–57].

Remark. If $m\leqslant 9$ then

$$P_m\cap Q_m=\begin{cases}\varnothing & \text{if } m \text{ is odd}\\ \{-1\} & \text{if } m \text{ is even.}\end{cases}$$

Thus in this case (1.1) is solvable if and only if m is odd and (1.3) is solvable if and only if m is even.

3. Construction of Spline Solutions

In this section we develop some new recursion relations, which can be used in order to construct the solutions of the Birkhoff problems (1.1) and (1.3). Denote by

$$p_i(x)=\sum_{\nu=0}^{m} a_{i\nu}(x-x_i)^{m-\nu} \qquad (i=0, ..., N) \tag{3.1}$$

the restriction of such a solution $s\in S_m(\Delta)$ on the interval $[x_i, x_{i+1}]$. For brevity we set

$$\left.\begin{aligned} p_i^{(\nu)} &:= p_i^{(\nu)}(x_i) \qquad (\nu=0, ..., m-1)\\ p_i &:= p_i^{(0)}=p_i(x_i)\end{aligned}\right\} \qquad (i=0, ..., N). \tag{3.2}$$

First Case: *m Odd*

First we shall express the coefficients $a_{i\nu}$ ($\nu = 0, ..., m$; $i = 0, ..., N$) by the numbers $p_i^{(\nu)}$ ($\nu = 0, 2, 4, ..., m-1$). If ν is odd, then obviously

$$a_{i\nu} = \frac{1}{(m-\nu)!} p_i^{(m-\nu)} \qquad (\nu = 1, 3, ..., m). \tag{3.3}$$

The other coefficients can be computed by the following lemma.

LEMMA 3.1. *Let the coefficients $c_{k,1}$ and $c_{k,0}$ be defined by*

$$c_{0,1} = 1 \qquad c_{2,1} = -\frac{1}{3!} \qquad c_{4,1} = -\frac{1}{5!} + \frac{1}{3!^2}$$

$$c_{k,1} = -\frac{c_{0,1}}{(k+1)!} - \frac{c_{2,1}}{(k-1)!} - \cdots - \frac{c_{k-2,1}}{3!}$$

$$c_{k,0} = -\frac{c_{0,1}}{k!} - \frac{c_{2,1}}{(k-2)!} - \cdots - \frac{c_{k-2,1}}{2!} - c_{k,1}.$$

Then

$$\begin{aligned} a_{i,m-\nu} = {} & \frac{1}{\nu! \cdot h} (c_{0,1} p_{i+1}^{(\nu-1)} + c_{0,0} p_i^{(\nu-1)}) \\ & + \frac{h}{\nu!} (c_{2,1} p_{i+1}^{(\nu+1)} + c_{2,0} p_i^{(\nu+1)}) \\ & + \cdots + \frac{h^{m-\nu-1}}{\nu!} (c_{m-\nu,1} p_{i+1}^{(m-1)} \\ & + c_{m-\nu,0} p_i^{(m-1)}) \\ & \qquad (\nu = 1, 3, ..., m; i = 0, ..., N). \end{aligned}$$

The proof of Lemma 3.1 *can be easily carried out by expanding $p_{i+1}^{(\nu)}$ on the right-hand side in a Taylor series.*

LEMMA 3.2. *With the above introduced constants $c_{k,1}$ and $c_{k,0}$ the following equations hold*:

$$\begin{aligned} & \frac{1}{h} (c_{0,1} p_{i+1}^{(\nu-1)} + 2c_{0,0} p_i^{(\nu-1)} + c_{0,1} p_{i-1}^{(\nu-1)}) \\ & + h(c_{2,1} p_{i+1}^{(\nu+1)} + 2c_{2,0} p_i^{(\nu+1)} + c_{2,1} p_{i+1}^{(\nu+1)} \\ & + \cdots + h^{m-\nu-1}(c_{m-\nu,1} p_{i+1}^{(m-1)} + 2c_{m-\nu,0} p_i^{(m-1)} \\ & + c_{m-\nu,1} p_{i-1}^{(m-1)}) = 0 \qquad (\nu = 1, 3, ..., m-2; i = 1, ..., N). \end{aligned}$$

Proof. Because of $p_{i-1}^{(\nu)}(x_i) = p_i^{(\nu)}(x_i)$ $(i = 1, ..., N)$ we have

$$\sum_{j=0}^{m-\nu} \frac{(m-j)!}{(m-j-\nu)!} h^{m-j-\nu} a_{i-1,j} = \nu!\, a_{i,m-\nu} \qquad (\nu = 1, 3, ..., m-2). \quad (3.4)$$

Now if the coefficients $a_{i-1,j}$ and $a_{i,j}$ in (3.4) are replaced by the corresponding terms in (3.3) then we obtain equations of the type

$$\sum_{\substack{0 \leqslant j \leqslant m-\nu \\ j \text{ even}}} h^{j-1}(c_{j,1} p_{i+1}^{(\nu+j-1)} + 2d_{j,0} p_i^{(\nu+j-1)} + d_{j,1} p_{i-1}^{(\nu+j-1)}) = 0$$

$$(\nu = 1, 3, ..., m-2;\ i = 1, ..., N). \quad (3.5)$$

It remains to show that

$$d_{j,1} = c_{j,1}, \qquad d_{j,0} = c_{j,0} \qquad (j = 0, 2, ..., m-1). \quad (3.6)$$

In order to prove $d_{j,1} = c_{j,1}$ we first remark that (3.5) is also valid after interchanging $c_{j,1}$ and $d_{j,1}$, because (3.5) can be applied on the "reflected" spline $\tilde{s}(x) = s(2x_i - x)$. Subtracting both equations leads to

$$\sum_{\substack{0 \leqslant j \leqslant m-\nu \\ j \text{ even}}} h^{j-1} e_j (p_{i+1}^{(\nu+j-1)} - p_{i-1}^{(\nu+j-1)}) = 0 \qquad (\nu = 1, 3, ..., m-2), \quad (3.7)$$

where $e_j = c_{j,1} - d_{j,1}$ $(j = 0, 2, ..., m-\nu)$.

Expanding (3.7) in a Taylor series at x_{i-1} it follows that $e_j = 0$. The proof of $d_{j,0} = c_{j,0}$ can be carried out by induction on j: Obviously we have $d_{0,0} = -c_{0,1} = c_{0,0}$. Assume that $d_{j-2,0} = c_{j-2,0}$. By expanding (3.5) in a Taylor series again it follows that

$$\begin{aligned} d_{j,0} &= -c_{j,1} - \frac{2}{2!} c_{j-2,1} - \frac{1}{2!} c_{j-1,0} - \cdots - \frac{2^{j-1}}{j!} c_{0,1} - \frac{1}{j!} c_{0,0} \\ &= -c_{j,1} - \left(\frac{2}{2!} - \frac{1}{2!}\right) c_{j-2,1} - \left(\frac{2^3}{4!} - \frac{1}{2!2!} - \frac{1}{4!}\right) c_{j-4,1} - \cdots \\ &\quad - \left(\frac{2^{j-1}}{j!} - \frac{1}{2!(j-2)!} - \frac{1}{4!(j-4)!} - \cdots - \frac{1}{j!}\right) c_{0,1} \\ &= -c_{j,1} - \frac{1}{2!} c_{j-2,1} - \cdots - \frac{1}{j!} c_{0,1} = c_{j,0} \end{aligned}$$

which completes the proof of Lemma 3.2. ∎

The equations of Lemma 3.1 and Lemma 3.2 can be used for computing all coefficients $a_{i\nu}$ of p_i ($i=0, ..., N$), if problem (1.1) is solvable.

EXAMPLE. For $m=5$ we obtain the following formulas, which were already developed by Usmani [7, p. 157]

$$7p_{i+1}^{(4)}+16p_i^{(4)}+7p_{i-1}^{(4)} = 60(p_{i+1}^{(2)}+4p_i^{(2)}+p_{i-1}^{(2)})/h^2-360(p_{i+1}-2p_i+p_{i-1})/h^4 \quad (3.8)$$

$$p_{i+1}^{(4)}+4_{p_i}^{(4)}+p_{i-1}^{(4)} = 6(p_{i+1}^{(2)}-2p_i^{(2)}+p_{i-1}^{(2)})/h^2 \qquad (i=1, ..., N). \quad (3.9)$$

Second Case: m Even

In this case it is not possible to express $a_{i\nu}$ as linear combinations of $p_i^{(\nu)}$ (ν even). Additionally one of the values $p_i^{(\nu)}$ (ν odd; $i=0, ..., N+1$) is needed. The following lemma contains an analogous statement like Lemma 3.1 and can be easily proved by Taylor's formula.

LEMMA 3.3. *Let* $i\in\{0, ..., N\}$. *Then*

$$a_{i,m-\nu}=\frac{p_i^{(\nu)}}{\nu!} \qquad (\nu=0, 2, 4, ..., m-2, m-1)$$

$$a_{i,m-\nu}=\frac{1}{\nu!}\sum_{\substack{0\leq k\leq m-\nu-3\\ k\text{ even}}} h^{k-1}(c_{k,1}p_{i+1}^{(\nu+k-1)}+c_{k,0}p_i^{(\nu+k-1)})$$
$$+\frac{h^{m-\nu-2}}{\nu!}(\tilde{c}_{m-\nu-1,1}p_{i+1}^{(m-2)}+\tilde{c}_{m-\nu-1,0}p_i^{(m-2)})$$
$$+\frac{h^{m-\nu-1}}{\nu!}\tilde{c}_{m-\nu,0}p_i^{(m-1)} \qquad (i=0, ..., N;\ \nu=1, 3, ..., m-3)$$

$$a_{i,0}=\frac{2}{m!\,h^2}(p_{i+1}^{(m-2)}-p_i^{(m-2)})-\frac{2}{m!\,h}p_i^{(m-1)},$$

where the coefficients $\tilde{c}_{k,1}$, $\tilde{c}_{k+1,0}$, and $\tilde{c}_{k,0}$ are recursively defined by

$$\tilde{c}_{k,1}=-2\left(\frac{c_{k-2,1}}{4!}+\frac{c_{k-4,1}}{6!}+\cdots+\frac{c_{0,1}}{(k+2)!}\right)$$

$$\tilde{c}_{k+1,0}=-\tilde{c}_{k,1}-\frac{c_{k-2,1}}{3!}-\cdots-\frac{c_{0,1}}{(k+1)}$$

$$\tilde{c}_{k,0}=-\tilde{c}_{k,1}-\frac{c_{k-2,1}}{2!}-\cdots-\frac{c_{0,1}}{k!} \qquad (k=m-2, m-4, ..., 0).$$

LEMMA 3.4. *The following recursion relations hold*:

$$\sum_{\substack{0\le k\le m-3\\ k \text{ even}}} h^{k-1}(c_{k,1}p_{i+1}^{(\nu+k-1)}+2c_{k,0}p_i^{(\nu+k-1)}+c_{k,1}p_{i-1}^{(\nu+k-1)})$$
$$+h^{m-\nu-2}(\tilde c_{m-\nu-1,1}p_{i+1}^{(m-2)}+\gamma_{m-\nu}p_i^{(m-2)}+\beta_{m-\nu}p_{i-1}^{(m-2)})$$
$$+h^{m-\nu-1}(\tilde c_{m-\nu,0}p_i^{(m-1)}+\alpha_{m-\nu}p_{i-1}^{(m-1)})=0$$
$$(\nu=1,3,...,m-3;\ i=1,...,N)$$

$$p_i^{(m-1)}+p_{i-1}^{(m-1)}-\frac{2}{h}(p_i^{(m-2)}-p_{i-1}^{(m-2)})=0 \qquad (i=1,...,N+1), \quad (3.10)$$

where $\alpha_\nu := \tilde c_{\nu,0}$, $\beta_\nu := \tilde c_{\nu-1,1}+2\tilde c_{\nu,0}$,

$$\gamma_\nu := -\sum_{\substack{0\le k\le \nu-3\\ k \text{ even}}}\left(\frac{2^{\nu-k-1}}{(\nu-k-1)!}c_{k,1}+\frac{2}{(\nu-k-1)!}c_{k,0}\right)$$
$$-2\tilde c_{\nu-1,1}-2\tilde c_{\nu,0} \qquad (\nu=3,...,m-1).$$

Proof. It can be carried out similarly as the proof of Lemma 3.2. We consider the equalities $p_{i-1}^{(\nu)}(x_i)=p_i^{(\nu)}(x_i)$ and replace the coefficients a_{ij} by the terms calculated in Lemma 3.3. Note that for any m the coefficients a_{ij} could be represented as linear combinations of $p_{i+1}^{(k)}$, $p_i^{(k)}$ $(k=0, 2, ...,)$ and that in both cases (Lemma 3.1 and Lemma 3.3) the coefficients $c_{k-\nu+1,1}$, $c_{k-\nu+1,0}$ corresponding to $p_{i+1}^{(k)}$, $p_i^{(k)}$ are the same for $k\le m-4$. Thus in the equations of Lemma 3.2 and Lemma 3.4 the same coefficients of $p_{i+1}^{(k)}$, $p_i^{(k)}$ occur for $k\le m-4$. The coefficients $\tilde c_{m-\nu-1,1}$ and $\tilde c_{m-\nu,0}$ corresponding to $p_{i+1}^{(m-2)}$ and $p_i^{(m-1)}$ in (3.10) also follow from $p_{i-1}^{(\nu)}(x_i)=p_i^{(\nu)}(x_i)$ $(\nu=1,3,...,m-1)$.

It remains to prove the equations for α_ν, β_ν, and γ_ν. Replacing s by $\tilde s(x):=s(2x_i-x)$ in (3.10) and taking into consideration that $\tilde s_{i-1}^{(\nu)}=(-1)^\nu s_{i+1}^{(\nu)}$ yields

$$\sum_{\substack{0\le k\le m-3\\ k \text{ even}}} h^{k-1}(c_{k,1}p_{i+1}^{(\nu+k-1)}+2c_{k,0}p_i^{(\nu+k-1)}+c_{k,1}p_{i-1}^{(\nu+k-1)})$$
$$+h^{m-\nu-2}(\beta_{m-\nu}p_{i+1}^{(m-2)}+\gamma_{m-\nu}p_i^{(m-2)}+\tilde c_{m-\nu-1,1}p_{i-1}^{(m-2)})$$
$$-h^{m-\nu-1}(\alpha_{m-\nu}p_{i+1}^{(m-1)}+\tilde c_{m-\nu,0}p_i^{(m-1)})=0 \qquad (\nu=1,3,...,m-3). \quad (3.11)$$

Subtracting (3.11) from (3.10) and dividing by $h^{m-\nu-2}$ we obtain

$$(\tilde c_{m-\nu-1,1}-\beta_{m-\nu})(p_{i+1}^{(m-2)}-p_{i-1}^{(m-2)})$$
$$+h(\alpha_{m-\nu}p_{i+1}^{(m-1)}+2\tilde c_{m-\nu,0}p_i^{(m-1)}+\alpha_{m-\nu}p_{i-1}^{(m-1)})=0$$
$$(\nu=1,3,...,m-3;\ i=1,...,N). \quad (3.12)$$

We now replace i by $i+1$ and use Taylor's formula for splines of degree m:

$$p_{i+k}^{(l)} = p_i^{(l)} + kh\, p_i^{(l+1)} + \cdots + \frac{k^{m-l-1}}{(m-l-1)!} h^{m-l-1} p_i^{(m-1)}$$
$$+ \frac{h^{m-1}}{(m-l)!} \sum_{\nu=0}^{k-1} ((k-\nu)^{m-l} - (k-\nu-1)^{m-1})\, p_{i+\nu}^{(m)}, \qquad (3.13)$$

where $p_{i+\nu}^{(m)} := \lim_{\varepsilon \searrow 0} s^{(m)}(x_{i+\nu} + \varepsilon)$. Therefore

$$2(-\beta_{m-\nu} + \tilde{c}_{m-\nu-1,1} + \alpha_{m-\nu} + \tilde{c}_{m-\nu,0})\, p_i^{(m-1)}$$
$$+ \frac{h}{2}(2\alpha_{m-\nu} - 3\beta_{m-\nu} + 3\tilde{c}_{m-\nu-1,1} + 4\tilde{c}_{m-\nu,0})\, p_i^{(m)}$$
$$+ \frac{h}{2}(2\alpha_{m-\nu} - \beta_{m-\nu} + \tilde{c}_{m-\nu-1,1})\, p_{i+1}^{(m)} = 0 \qquad (\nu = 1, 3, ..., m-3). \qquad (3.14)$$

Because of the uniqueness of the Taylor coefficients the terms in brackets must be zero, hence

$$\alpha_{m-\nu} = \tilde{c}_{m-\nu,0}$$
$$\beta_{m-\nu} = \tilde{c}_{m-\nu-1,1} + 2\tilde{c}_{m-\nu,0} \qquad (\nu = 1, 3, ..., m-3).$$

In the same manner the equation for $\gamma_{m-\nu}$ can be derived. Thus the proof of Lemma 3.4 is complete. ▮

Given p_i and $p_i^{(2)}$ Lemma 3.4 can be used for computing all even derivatives $p_i^{(\nu)}$ $(\nu = 0, 2, ..., m-2)$. In order to calculate $p_i^{(m-1)}$ $(i = 1, ..., N)$ $p_0^{(m-1)}$ must be fixed. For example one can demand

$$p_0^{(m-1)} = \sum_{\substack{0 \leqslant k \leqslant m-4 \\ k \text{ even}}} h^{k-m+1}(c_{k,1}\, p_2^{(k)} + 2c_{k,0}\, p_1^{(k)} + c_{k,1}\, p_0^{(k)})$$
$$+ (c_{m-2,1}\, p_2^{(m-2)} + (c_{m-2,0} + 1)\, p_1^{(m-2)}$$
$$+ (c_{m-2,1} - 1)\, p_0^{(m-2)})/h. \qquad (3.15)$$

EXAMPLE. $m = 6$. Let

$$p_i(x) = \sum_{\nu=0}^{6} a_{i,6-\nu}(x - x_i)^\nu$$
$$y_i := p_i,\ M_i := p_i^{(2)},\ S_i = p_i^{(4)},\ F_i = p_i^{(5)}.$$

Then the coefficients a_{ij} of p_i are given by

$$a_{i,0}=\frac{1}{360h^2}(S_{i+1}-S_i)-\frac{1}{360h}F_i; \qquad a_{i,1}=\frac{1}{120}F_i;$$

$$a_{i,2}=\frac{1}{24}S_i; \qquad a_{i,3}=\frac{1}{6h}(M_{i+1}-M_i)-\frac{h}{72}(S_{i+1}+5S_i)-\frac{h^2}{72}F_i;$$

$$a_{i,4}=\frac{1}{2}M_i;$$

$$a_{i,5}=\frac{1}{h}(y_{i+1}-y_i)-\frac{h}{6}(M_{i+1}+2M_i) + \frac{h^3}{360}(4S_{i+1}+11S_i)+\frac{h^4}{120}F_i;$$

$$a_{i,6}=y_i; \qquad (i=0, ..., N).$$

The formulas of Lemma 3.4 are

$$\frac{1}{h}(y_{i+1}-2y_i+y_{i-1})-\frac{h}{6}(M_{i+1}+4M_i+M_{i-1}) + \frac{h^3}{360}(4S_{i+1}+16S_i+10S_{i-1})+\frac{h^4}{120}(F_i+F_{i-1})=0 \qquad (3.16)$$

$$\frac{1}{h}(M_{i+1}-2M_i+M_{i-1})-\frac{h}{12}(S_{i+1}+8S_i+3S_{i-1}) - \frac{h^2}{12}(F_i+F_{i-1})=0 \qquad (3.17)$$

$$\frac{2}{h}(S_i-S_{i-1})-(F_i+F_{i-1})=0 \qquad (i=1, ..., N). \qquad (3.18)$$

We insert formula (3.18) in (3.16) and (3.17) and obtain

$$\frac{1}{h}(y_{i+1}-2y_i+y_{i-1})-\frac{h}{6}(M_{i+1}+4M_i+M_{i-1}) + \frac{h^3}{360}(4S_{i+1}+22S_i+4S_{i-1})=0 \qquad (3.19)$$

$$\frac{1}{h}(M_{i+1}-2M_i+M_{i-1})-\frac{h}{12}(S_{i+1}+10S_i+S_{i-1})=0 \qquad (i=1, ..., N). \qquad (3.20)$$

Hence (see also Usmani [7, p. 160])

$$S_i=\frac{20}{h^4}(y_{i+1}-2y_i+y_{i-1})-\frac{2}{3h^2}(M_{i+1}+28M_i+M_{i-1}) \qquad (i=1, ..., N). \tag{3.21}$$

S_0 and S_{N+1} can be computed from formula (3.20). F_0 is given by (3.15),

$$F_0=\frac{1}{h^5}(y_2-2y_1+y_0)-\frac{1}{6h^3}(M_2+4M_1+M_0)$$
$$+\frac{1}{360h}(7S_2+376S_1-353S_0),$$

and F_i ($i=1, ..., N$) by (3.18). Thus all values for computing the coefficients a_{ij} are known.

4. Application to Spline Collocation

We give an example, where the Hermite–Birkhoff problems (1.1) and (1.3) play an important role.

Consider the two-point boundary value problem

$$\begin{aligned} y''(x)&=f(x, y(x))\\ y(a)&=A\\ y(b)&=B, \end{aligned} \tag{4.1}$$

where f is a real-valued bivariate function and A, B are given real numbers. A frequently used method for solving (4.1) numerically is the method of collocation: A spline function $s\in S_m(\Delta)$ is looked for such that

$$\begin{aligned} s''(x_i)&=f(x_i, s(x_i)) \qquad i=0, ..., N+1\\ s(a)&=A\\ s(b)&=B. \end{aligned} \tag{4.2}$$

In [4] we showed that discrete values $s(x_i)$ $(i=0, ..., N+1)$ can be obtained by the solution of the nonlinear system of equations

$$\sum_{\mu=1}^{m} b_{m,\mu}^{(2)} s(x_{\mu+\nu-1})=\sum_{\mu=1}^{m} b_{m,\mu} f(x_{\mu+\nu-1}, s(x_{\mu+\nu-1}))$$
$$(\nu=0, ..., N-m+2)$$

if m is odd and

$$\sum_{\mu=1}^{m-1} \sum_{\sigma=1}^{\mu} (-1)^{\mu-\sigma} b_{m,\sigma}^{(2)} s(x_{\mu+\nu-1})$$
$$= \sum_{\mu=1}^{m-1} \sum_{\sigma=1}^{\mu} (-1)^{\mu-\sigma} b_{m,\sigma} f(x_{\mu+\nu-1}, s(x_{\mu+\nu-1}))$$
$$(\nu=0, ..., N-m+1)$$

if m is even. If u denotes the exact solution of (4.1), then the speed of convergence is given by

$$\max_{1 \leqslant i \leqslant N} |u(x_i)-s(x_i)| = O(h^{\hat{m}}),$$

where $\hat{m}=m$ if m is even and $\hat{m}=m-1$ if m is odd.

Now a global solution of (4.2) (i.e., the coefficients of a basis) can be calculated by the methods of Section 3. Furthermore it can be shown that the approximate solution s converges uniformly to the exact solution u of (4.1) with the following rate of convergence:

$$\max_{x \in [a,b]} |u^{(\nu)}(x)-s^{(\nu)}(x)| = O(h^{m-\nu-1}) \qquad (\nu=0, ..., m-1).$$

For details, see [4, p. 70–79].

CONCLUDING REMARK

It is quite easy to generalize the results of Sections 2 and 3 in order to solve Birkhoff problems of the type

$$s(x_i)=y_i$$
$$s^{(k)}(x_i)=y_i^{(k)}$$
$$s \in S_m(\Delta),$$

where y_i, $y_i^{(k)}$ are given real numbers and k is a given positive integer. These generalizations can be used for solving boundary value problems of the form $y^{(k)}(x)=f(x, y(x))$. Special cases are treated in the papers of Isa and Usmani [3] and Usmani [6].

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